

ON VALUATION RINGS

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ABSTRACT. In this paper we provide necessary and sufficient conditions for $R = A \ltimes E$ to be a valuation ring where E is a non-torsion or finitely generated A -module. Also, we investigate the (n, d) property of the valuation ring.

1. INTRODUCTION

All rings considered will be commutative and have identity element; all modules will be unital.

Let A be a ring, an A -module E is said to be *uniserial* if the set of its submodules is totally ordered by inclusion; equivalently, for every $(x, y) \in E^2$, $x \in Ay$ or $y \in Ax$. A ring A is called a *valuation ring* if A is an uniserial A -module. We note that A is a valuation ring if and only if A is a local ring and every finitely generated ideal is principal. See for instance [[5], [6], [8], [12], [13], [21]].

An *arithmetical ring* is a ring A for which the ideals form a distributive lattice, i.e for which $(\mathfrak{a} + \mathfrak{b}) \cap \mathfrak{c} = (\mathfrak{a} \cap \mathfrak{c}) + (\mathfrak{b} \cap \mathfrak{c})$ for all ideals of A . In [13] C.U. Jensen gives some more characterization of arithmetical ring, it is proved that a ring A is an arithmetical ring if and only if every localization $A_{\mathfrak{m}}$ at a maximal (prime) ideal \mathfrak{m} is a valuation ring. See for instance [[1], [2], [5], [6], [8], [9], [13]].

Let A be a ring, E be an A -module and $R := A \ltimes E$ be the set of pairs (a, e) with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$. R is called the trivial ring extension of A by E (also called the idealization of E over A). Considerable work, part of it summarized in Glaz's book [9] and Huckaba's book [12], has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and non-commutative) ring theory. See for instance [[1], [2], [9], [12], [14], [15], [19]].

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For nonnegative integer n , an A -module E is said to be of finite n -presentation (or n -presented) if there exists an exact sequence:

$$F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow E \rightarrow 0$$

where F_i is a free A -module of finite rank. We write

$$\lambda_A(E) = \sup \{n; \text{there exists a finite } n\text{-presentation of } E\}.$$

The λ -dimension of a ring A ($\lambda - \dim A$) is the least integer n (or ∞ if none such exists) such that $\lambda_A(E) \geq n$ implies $\lambda_A(E) = \infty$. A is called a strong n -coherent ring in [[14], [15], [16]], [17]].

Throughout, $\text{pd}_A(E)$ will denote the projective dimension of E as an A -module.

Given nonnegative integers n and d , we say that a ring R is an (n, d) -ring if $\text{pd}_R(E) \leq d$ for each n -presented R -module E (as usual, pd denotes projective dimension). For integers $n, d \geq 0$ Costa asks in [3] whether there is an (n, d) -ring which is neither an $(n, d-1)$ -ring nor an $(n-1, d)$ -ring? The answer is affirmative for $(0, d)$ -rings, $(1, d)$ -rings, $(2, d)$ -rings and $(3, d)$ -rings for each integer d . See for instance [[3], [4], [14], [15], [16]], [17], [22]].

The goal of section 2 of this paper is to provide necessary and sufficient conditions for $R := A \ltimes E$ to be a valuation ring where E is a non-torsion or finitely generated A -module. The section 3 is devoted to investigate the (n, d) -property of the valuation ring.

2. TRIVIAL EXTENSIONS DEFINED BY VALUATION RING

This section develops a result of the transfer of valuation property to trivial ring extension. Recall that an A -module E is called a torsion module if for every $u \in E$, there exists $0 \neq a \in A$ such that $au = 0$.

Theorem 2.1. *Let A be a ring and E an nonzero A -module. Let $R := A \ltimes E$ be the trivial ring extension of A by E .*

- (1) *Assume that E is a non-torsion A -module. Then R is a valuation ring if and only if A is a valuation domain and E is isomorphic to $K := \text{qf}(A)$, the field of fractions of A .*
- (2) *Assume that E is a finitely generated A -module. Then R is a valuation ring if and only if A is field and $E \simeq A$.*

Before proving Theorem 2.1, we establish the following Lemma:

Lemma 2.2. *Let A be a ring, E a non zero A -module and let $R := A \ltimes E$ be the trivial ring extension of A by E . If R is a valuation ring then A is a valuation domain and E is an uniserial A -module.*

Proof. Assume that R is a valuation ring. First we wish to show that A is a valuation ring and E is a uniserial A -module. Let $(a, b) \in A^2$, if $(a, 0)$ divides $(b, 0)$ (resp., $(b, 0)$ divides $(a, 0)$) then a divides b (resp., b divides a). Hence A is a valuation ring. On the other hand, let $(x, y) \in E^2$. If $(0, x)$ divides $(0, y)$ (resp., $(0, y)$ divides $(0, x)$) then there exists $(c, z) \in R$ such that $(0, y) = (c, z)(0, x)$ (resp., $(0, x) = (c, z)(0, y)$) and so $y \in Ax$ (resp., $x \in Ay$). Therefore E is an uniserial A -module.

We claim that A is an integral domain. Deny. Let $(a, b) \in A^2$ such that $ab = 0$, $a \neq 0$ and $b \neq 0$. For each $x \in E$, $(b, 0)$ divides $(0, x)$ (since R is a valuation ring and $(0, x)$ does not divide $(b, 0)$ (since $b \neq 0$)) and so there exists $y \in E$ such that $by = x$, thus $ax = 0$ and so $a \in (0 : E)$. Also, for each $x \in E$, $(a, 0)$ divides $(0, x)$ and so $x \in aE = 0$, a contradiction since $E \neq 0$. Thus A is an integral domain.

□

Proof. of Theorem 2.1.

1) Assume that A is a valuation domain and let $R := A \propto K$, where $K := qf(A)$. Our aim is to show that R is a valuation ring. Let $(a, x), (b, y) \in R - \{0, 0\}$. Two cases are then possibles:

Case 1. $a = b = 0$. There exists then $c \in A$ such that $x = cy$ (resp., $y = cx$) since $K := qf(A)$ and A is a valuation domain. Hence, $(0, x) = (c, 0)(0, y)$ (resp., $(0, y) = (c, 0)(0, x)$) as desired.

Case 2. $a \neq 0$ or $b \neq 0$. We may assume that $a \neq 0$ and $b \in Aa$. Let $c \in A$ such that $ac = b$ and let $z \in K$ such that $az + cx = y$. Hence, $(a, x)(c, z) = (b, y)$ as desired.

Conversely, assume that E is a non-torsion A -module and R is a valuation ring. We wish to show that $E \simeq K$. Let $u \in E$ such that $(0 : u) = 0$ and let $f : K \otimes Au \rightarrow K \otimes E$ be the homomorphism of A -module induced by the inclusion map $Au \hookrightarrow E$. Since the field K is a flat A -module, then f is injective. Let $(\lambda, x) \in K \times E$, by Lemma 2.2 we get that $x \in Au$ or $u \in Ax$. If $x = au$ for some $a \in A$ then $f(\lambda \otimes au) = \lambda \otimes x$. If $u \in Ax$ then there exists $a \in A$ such that $u = ax$. Thus

$$f\left(\frac{\lambda}{a} \otimes u\right) = \frac{\lambda}{a} \otimes u = \frac{\lambda}{a} \otimes ax = \lambda \otimes x.$$

Consequently, f is an isomorphism of A -module. Now, consider the homomorphism of A -module $g : E \rightarrow K \otimes E$ defined by $g(x) = 1 \otimes x$. For all multiplicatively closed subset S of A , the $S^{-1}A$ -modules $S^{-1}E$ and $S^{-1}A \otimes_A E$ are isomorphic; more precisely the map $\varphi : S^{-1}E \rightarrow S^{-1}A \otimes_A E$, where $\varphi\left(\frac{x}{s}\right) = \frac{1}{s} \otimes x$ is isomorphism. If $g(x) = 1 \otimes x = 0$, then there exists $0 \neq a \in A$ such that $ax = 0$. By Lemma 2.2 $x \in Au$ or $u \in Ax$. But

$u \notin Ax$ since $ax = 0$, $a \neq 0$ and $(0 : u) = 0$. Hence, $x = bu$ for some $b \in A$. Then $abu = 0$, hence $ab = 0$ since $(0 : u) = 0$ and so $b = 0$ (since A is a valuation domain and $a \neq 0$); thus $x = 0$. It follows that g is injectif. Let $(\lambda, x) \in K \times E$, if $\lambda \in A$ then $\lambda \otimes x = 1 \otimes \lambda x = g(\lambda x)$. Now if $\lambda^{-1} \in A$ then there exists $y \in E$ such that $\lambda^{-1}y = x$, since $(\lambda^{-1}, 0)$ divides $(0, x)$. Hence

$$\lambda \otimes x = \lambda \otimes (\lambda^{-1}y) = 1 \otimes y = g(y).$$

Consequently, g is an isomorphism of A -module. We deduce that

$$E \simeq K \otimes_A E \simeq K \otimes_A Au \simeq K \otimes_A A \simeq K.$$

2) If A is a field, then $R := A \propto A$ is a valuation ring by the proof of 1) above. Conversely, assume that E is a finitely generated A -module. We denote by \mathfrak{m} the maximal ideal of A . By Lemma 2.2, $E/\mathfrak{m}E$ is an A/\mathfrak{m} -vector space and for all $(x, y) \in E^2$, $\bar{x} \in (A/\mathfrak{m})\bar{y}$ or $\bar{y} \in (A/\mathfrak{m})\bar{x}$. Hence, $\dim_{A/\mathfrak{m}A}(E/\mathfrak{m}E) = 0$ or 1 . If $E = \mathfrak{m}E$, then $E = 0$ by Nakayama Lemma which is absurd. Thus $E/\mathfrak{m}E = (A/\mathfrak{m})\bar{v}$ for some $v \in E \setminus \mathfrak{m}E$. By Nakayama Lemma v generate E . Suppose that $\mathfrak{m} \neq 0$, let $0 \neq a \in \mathfrak{m}$, we have $(a, 0)$ divides $(0, v)$, then there exists $b \in A$ such that $(a, 0)(0, bv) = (0, v)$. Hence $(1 - ab)v = 0$, therefore $v = 0$ (since $1 - ab$ is unit), which is absurd. Therefore, the ring A is a field and $E = Av$, therefore $E \simeq A$ completing the proof of Theorem 2.1. \square

Corollary 2.3. *Let A be a ring, then $A \propto A$ is a valuation ring if and only if A is a field.*

Theorem 2.1 enriches the literature with new examples of valuation rings, as shown below.

Example 2.4. Let k be a field. Let $A = k[[x]]$ the ring of formal power series with coefficients in k and $K = k((x))$ its field of fractions. The trivial ring extension of A by K , $A \propto K$ is a valuation ring.

Example 2.5. Let \mathbb{Q}_p be the completion of \mathbb{Q} in the p -adic topology where p is prime integer. The ring of p -adic integers is $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$, \mathbb{Q}_p is its field of fractions. Then the trivial ring extension of \mathbb{Z}_p by \mathbb{Q}_p is a valuation ring.

Now, we are able to construct a non-valuation ring.

Corollary 2.6. *Let A be a ring and let E be a mixed module, i.e E is neither torsion nor torsion-free. Then $A \propto E$ is not a valuation ring.*

Proof. Assume that $A \propto E$ is a valuation ring and that E is a non torsion A -module. Let $u \in E$ such that $(0 : u) = 0$. By Lemma 2.2, for each $0 \neq x \in E, x \in Au$ or $u \in Ax$. Suppose that $u = ax$ for some $a \in A$. Since $(0 : u) = 0$, the following implications hold:

$$\alpha x = 0 \Rightarrow \alpha ax = 0 \Rightarrow \alpha u = 0 \Rightarrow \alpha = 0.$$

Hence $(0 : x) = 0$. Finally, it is easy to get the equality $(0 : x) = 0$ in the case $x \in Au$ since A is an integral domain (by Lemma 2.2). Thus E is a torsion-free A -module. \square

3. (n, d) -PROPERTIES AND VALUATION RINGS

Let A be a ring. An A -module is called a cyclically presented module if it is isomorphic to A/aA for some $a \in A$.

Now, we are able to give our main result in this section.

Theorem 3.1. *Let A be a valuation ring and let Z be the subset of its zero divisors.*

- (1) *If $(0 : a)$ is not a finitely generated ideal for every $a \in Z \setminus 0$, then A is a $(2, 1)$ -ring.*
- (2) *If $(0 : a)$ is a finitely generated ideal for some $a \in Z \setminus 0$, then A is not a $(2, d)$ -ring for every nonnegative integer d .*

In order to prove Theorem 3.1, we will use the following Lemma:

Lemma 3.2. ([21, Theorem 1])

A finitely presented module over a valuation ring is a finite direct sum of cyclically presented modules.

Proof. of Theorem 3.1.

1) Assume that $(0 : a)$ is not a finitely generated ideal for every $a \in Z \setminus 0$ and let E be a 2-presented A -module. We want to show that $pd_A(E) \leq 1$. By the above Lemma 3.2, E is finite direct sum of cyclically presented modules; i.e $E = \bigoplus_{i=1}^n Ax_i$ and $Ax_i \simeq A/a_iA$ for some $a_i \in A$. Consider the following exact sequences:

$$0 \rightarrow a_iA \rightarrow A \rightarrow A/a_iA \rightarrow 0 \quad (1)$$

$$0 \rightarrow (0 : a_i) \rightarrow A \rightarrow a_iA \rightarrow 0. \quad (2)$$

Then a_iA is a finitely presented A -module and $(0 : a_i)$ is a finitely generated ideal (since A/a_iA is a 2-presented A -module). Therefore, $a_i = 0$ or a_i is a nonzero divisor element of A by hypothesis. Using the exact sequence (1), we can also deduce that $pd(A/a_iA) \leq 1$ and so $pd(E) =$

$\sup \{pd_A(A/a_iA) : 1 \leq i \leq n\} \leq 1$. Hence, A is a $(2, 1)$ -ring.

2) Let $a \in Z \setminus 0$ such that $(0 : a)$ is a finitely generated ideal. The following exact sequences of A -modules:

$$0 \rightarrow aA \rightarrow A \rightarrow A/aA \rightarrow 0 \quad (1)$$

$$0 \rightarrow (0 : a) \rightarrow A \rightarrow aA \rightarrow 0 \quad (2)$$

show that aA is a 1-presented A -module and A/aA is a 2-presented A -module. But the λ -dimension of every valuation ring is at most two ([5, Corollary 2.12]), hence $\lambda_A(A/aA) = \infty$. Let $b \in A$ such that $(0 : a) = bA$. By the following exact sequence of A -modules:

$$0 \rightarrow (0 : b) \rightarrow A \rightarrow bA \rightarrow 0 \quad (3)$$

we get that $(0 : b)$ is finitely generated. Then there exists $c \in A$ such that $(0 : b) = cA$. We claim that $(0 : c) = bA$.

Indeed, $bA \subseteq (0 : c)$ since $bc = 0$. On the other hand, let $x \in (0 : c)$, that is $cx = 0$. But $a \in (0 : b) = cA$ (since $(0 : a) = bA$), then $a = ct$ for some $t \in A$. Hence, $ax = cxt = 0$ and so $x \in (0 : a) = bA$, as desired.

By using the exact sequences (3) and

$$0 \rightarrow (0 : c) := bA \rightarrow A \rightarrow cA \rightarrow 0 \quad (4)$$

we get the equalities $pd_A(bA) = pd_A(cA) + 1$ and $pd_A(cA) = pd_A(bA) + 1$. Hence $pd_A(bA) = pd_A(cA) = \infty$. Therefore $pd_A(A/aA) = \infty$, which completes the proof of Theorem 3.1. \square

Now, we construct a $(2, 1)$ -ring which is a particular case of [15, Theorem 3.1].

Corollary 3.3. *Let A be a valuation domain which is not a field, $K := qf(A)$ and let $R := A \propto K$ be the trivial ring extension of A by K . Then R is a $(2, 1)$ -ring.*

Proof. By Theorem 2.1 R is a valuation ring. Let $0 \neq (a, x) \in R$. It is easy to get successively that (a, x) is a zero divisor if and only if $a = 0$, and the equality $(0 : (0, x)) = 0 \propto K$. Assume that there exists $0 \neq x \in K$ such that $(0 : (0, x))$ is not a finitely generated ideal. Then there exists $x_1, \dots, x_n \in K$ such that

$$(0 : (0, x)) = R(0, x_1) + \dots + R(0, x_n) = 0 \propto (Ax_1 + \dots + Ax_n).$$

Therefore, $K = Ax_1 + \dots + Ax_n$. We put $x_i = \frac{a_i}{d}$ for every $1 \leq i \leq n$, where $a_i \in A$ and $0 \neq d \in A$. Hence, $K = dK = Aa_1 + \dots + Aa_n \subseteq A$, a contradiction. Therefore, R is a $(2, 1)$ -ring. \square

Let A be a ring, a necessary and sufficient conditions for A to be coherent is that $(0 : a)$ is finitely generated ideal for every element $a \in A$, and the intersection of two finitely generated ideals of A is a finitely generated ideal of A . By Theorem 3.1, we have:

Corollary 3.4. *Let A be a valuation ring with zero divisors, if A is coherent then A is not a $(2, d)$ -ring for every nonnegative integer d .*

Remark 3.5. Let A be a valuation ring and \mathfrak{m} its maximal ideal. By [5, proposition 2.10], A is $(2, 1)$ -ring if and only if \mathfrak{m} is flat.

Example 3.6. Let p be a prime nonnegative integer and $n \in \mathbb{N}^*$. The valuation ring $\mathbb{Z}/p^n\mathbb{Z}$ is not a $(2, d)$ -ring for every nonnegative integer d .

Example 3.7. Let K be a field and n an integer such that $n \geq 2$. We denote $A = K[x]/(x^n)$ and $\overline{P} = P + A$ for every $P \in K[x]$. It is easy to see that A is a valuation ring. We have $(0 : \overline{x^{n-1}}) = \overline{x}A$, which is a finitely generated ideal. Hence A is not a $(2, d)$ -ring for every positive integer d .

Now, we study the relationship between the (n, d) -properties and an arithmetical rings.

Proposition 3.8. *Let A be an arithmetical ring.*

- (1) *Suppose that for every maximal ideal \mathfrak{m} of A and $0 \neq x \in Z(A_{\mathfrak{m}})$, $(0 : x)$ is not a finitely generated ideal. Then A is a $(3, 1)$ -ring.*
- (2) *If $(0 : x)$ is finitely generated for some maximal ideal \mathfrak{m} of A and $0 \neq x \in Z(A_{\mathfrak{m}})$, then A is not a $(1, d)$ -ring for every nonnegative integer d .*

Proof. (1) Let E be a 3-presented A -module and let \mathfrak{m} be a maximal ideal of A . The $A_{\mathfrak{m}}$ -module $E_{\mathfrak{m}}$ is then 2-presented. For every $x \in Z(A_{\mathfrak{m}})$, $(0 : x)$ is not finitely generated and $A_{\mathfrak{m}}$ is a valuation ring, then by Theorem 3.1 $A_{\mathfrak{m}}$ is $(2, 1)$ -ring. Thus the projective dimension of $E_{\mathfrak{m}}$ over $A_{\mathfrak{m}}$ is at most one. Since $\lambda - \dim A \leq 3$ (by [5, Theorem 2.1]), then E admits a finite free resolution. We deduce that

$$pd_A(E) = \sup \{pd_{A_{\mathfrak{m}}}(E_{\mathfrak{m}}) ; \mathfrak{m} \text{ is a maximal ideal of } A\} \leq 1$$

and so A is a $(3, 1)$ -ring.

(2) Assume that $(0 : x)$ is a finitely generated ideal for some maximal ideal \mathfrak{m} of A and $0 \neq x \in Z(A_{\mathfrak{m}})$. We put $x = \frac{a}{s}$ where $a \in A$ and

$s \notin \mathfrak{m}$. Then $pd_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/aA_{\mathfrak{m}}) = pd_{A_{\mathfrak{m}}}((A/aA)_{\mathfrak{m}})$ since the $A_{\mathfrak{m}}$ -modules $A_{\mathfrak{m}}/aA_{\mathfrak{m}}$ and $(A/aA)_{\mathfrak{m}}$ are isomorphic. Hence, $pd_{A_{\mathfrak{m}}}((A/aA)_{\mathfrak{m}}) = \infty$ by Theorem 3.1(2) and so $pd_A(M) \geq pd_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})$ for every A -module M , then $pd_A(A/aA) = \infty$. On the other hand, A/aA is a finitely presented A -module. Consequently, A is not a $(1, d)$ -ring for every nonnegative integer d . \square

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